The space of nodal curves of type p, q with given Weierstraß semigroup

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Abstract

We continue the investigation of curves of type p, q started in [KKW]. We study the space of such curves and the space of nodal curves with prescribed Weierstraß semigroup. A necessary and sufficient criterion for a numerical semigroup to be a Weierstraß semigroup is given. We find a class of Weierstraß semigroups which apparently has not yet been described in the literature.

Introduction

Let K be an algebraically closed field of characteristic 0. For relatively prime numbers $p, q \in \mathbb{N}$ with $1 < p < q$ a plane curve C of type p, q is the zero-set of a Weierstraß polynomial of type p, q

$$F(X,Y) := Y^p + bX^q + \sum_{\nu p + \mu q < pq} b_{\nu\mu} X^\nu Y^\mu \ (b_{\nu\mu} \in K, b \in K \setminus \{0\})$$

in $\mathbb{A}^2(K)$. Such curves are irreducible and have only one place P at infinity, i.e. P is the only point at infinity of the normalization of the projective closure $\mathcal{R}$ of C. The Weierstraß semigroup $H(P)$ of $\mathcal{R}$ at P is also called the Weierstraß semigroup of C. It contains the semigroup H_{pq} generated by p and q as a subsemigroup. Hence $H(P)$ is obtained from H_{pq} by closing some of its $d := {}^1\!/_2(p-1)(q-1)$ gaps. Remember that H_{pq} is a symmetric semigroup with conductor $c := (p-1)(q-1)$. It is shown in [KKW] that any Weierstraß semigroup is the Weierstraß semigroup of a plane curve of type p, q having only nodes as singularities if p and q are properly chosen.

By the substitution $X \mapsto 1/\sqrt[q]{-b} \cdot X, Y \mapsto Y$ the polynomial F goes over into a *normed* Weierstraß polynomial of type p, q

$$Y^p - X^q + \sum_{\nu p + \mu q < pq} a_{\nu\mu} X^\nu Y^\mu$$

whose zero-set it isomorphic to C and has the same place at infinity and the same Weierstraß semigroup. We call it the associated normed curve of C. In this paper we understand by curves of type p, q the plane curves defined by normed Weierstraß polynomials of type p, q.

These curves can be identified with the points $(\{a_{\nu\mu}\}_{\nu p + \mu q < pq}) \in \mathbb{A}^n(K)$ associated with their equation where $n := {}^1\!/_2(p+1)(q+1) - 1$. In Section 1 we describe the (locally closed)

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subsets of $\mathbb{A}^n(K)$ consisting of the various kinds of curves of type p, q. In particular we are interested in the set of nodal curves of type p, q. Such curves have at most d nodes, and their Weierstraß semigroup has genus $g = d - l$ if l is the number of the nodes. The singular nodal curves form a dense open set of an irreducible hypersurface $\mathcal{H} \subset \mathbb{A}^n(K)$ whose properties are the main object of study in Section 1. It turns out that for any $l \in \{0, \ldots, d\}$ a nodal curve of type p, q exists having exactly l nodes (Theorem 1.6).

Given a numerical semigroup H with $p \in H$ greater than the elements of a minimal system of generators of H we construct in Section 2 a locally closed subset $V_{pq}(H)$ in some affine space, such that H is a Weierstraß semigroup if and only if $V_{pq}(H) \neq \emptyset$ (Corollary 2.4). The set $V_{pq}(H)$ is explicitly described by polynomial vanishing and non-vanishing conditions, where "explicit" means that the polynomials are given by a formula or there is an algorithm to compute them. In principle the membership test for polynomial ideals allows then to decide whether H is a Weierstraß semigroup or not. However for any H of interest (i.e. where the result is not known) the number of conditions is huge so that the criterion seems only to be of theoretical interest and not feasible for a computer program.

By the *simplification of nodal curves* introduced in Section 3 the criterion allows to show without computations that every H of the following kind is a Weierstraß semigroup: H is obtained from the semigroup H_{pq} generated by p and q ($1 < p < q$, with p, q relatively prime) by closing all gaps of H_{pq} which are greater than or equal to a given gap of H_{pq} (Theorem 3.2). The *hyperordinary semigroups* defined by Rim and Vitulli [RV] belong to this class of semigroups. These authors have shown with a different method that hyperordinary semigroups are Weierstraß semigroups.

1 The space of plane curves of type p, q

Let $R := K[\{A_{\nu\mu}\}_{\nu p + \mu q < pq}]$ be the polynomial ring in the $n = {}^1\!/_2(p+1)(q+1) - 1$ indeterminates $A_{\nu\mu}$ ($\nu p + \mu q < pq$) over K. The generic (normed) Weierstraß polynomial

$$F = Y^p - X^q + \sum_{\nu p + \mu q < pq} A_{\nu\mu} X^\nu Y^\mu = A_{00} + \ldots$$

of type p, q has the partial derivatives

$$F_X = -qX^{q-1} + \sum_{\nu p + \mu q < pq} \nu A_{\nu\mu} X^{\nu-1} Y^\mu = A_{10} + \ldots$$

$$F_Y = pY^{p-1} + \sum_{\nu p + \mu q < pq} \mu A_{\nu\mu} X^\nu Y^{\mu-1} = A_{01} + \ldots$$

where the dots represent terms containing X or Y. We are interested in the ring

$$A = R[X, Y]/(F, F_X, F_Y)$$

as an R-Algebra. As a K-algebra it is isomorphic to the polynomial ring

$$K[\{A_{\nu\mu}\}_{(\nu,\mu) \neq (0,0),(1,0),(0,1)}, X, Y]$$

hence the image R' of R in A is a domain. Moreover $\{F, F_X, F_Y\}$ is a regular sequence in $R[X,Y]$.

We endow $R[X,Y]$ with the grading given by $\deg(X) = p$, $\deg(Y) = q$ and $\deg(r) = 0$ for $r \in R$ and let $\mathcal{F}$ denote the corresponding degree filtration. Let $N := R[X,Y]/(F_X, F_Y)$. The polynomial F has degree form $Y^p - X^q$, and since the partial derivatives are homogeneous maps the degree form of F_X is $-qX^{q-1}$ and that of F_Y is pY^{p-1}. Since they form a regular sequence in $R[X,Y]$ we have $\mathrm{gr}_{\mathcal{F}} N = R[X,Y]/(X^{q-1}, Y^{p-1})$ (see [Ku2], B.12), hence N is a free R-module with the basis

$$B := \{\xi^\nu \eta^\mu\}_{\nu < q-1, \mu < p-1}$$

where ξ, η are the residue classes of X, Y in N (see [Ku2], B.6). In particular $\mathrm{rank}(N) = (p-1)(q-1) =: c$, and different basis elements have different degrees with respect to the residue grading of $\mathcal{F}$.

Since A is finite over R' we have $R' = R/\mathfrak{q}$ where the prime ideal $\mathfrak{q}$ is generated by an irreducible polynomial in R, hence $\mathcal{H} := \mathrm{Spec}(R')$ is an irreducible hypersurface in $A^n(K) = \mathrm{Spec}(R)$.

We identify the curves of type p, q with the closed points $\alpha := (\{a_{\nu\mu}\}) \in \mathbb{A}^n(K)$ or with the maximal ideals $\mathfrak{m} = (\{A_{\nu\mu} - a_{\nu\mu}\}_{\nu p + \mu q < pq})$ $(a_{\nu\mu} \in K)$ of R. For $\alpha \in K^n$ we denote the curve with the equation $F(\alpha, X, Y) = 0$ by C_α. The set $\mathrm{Max}(A)$ can be identified with the set of singularities of the curves of type p, q. If a maximal ideal $\mathfrak{M}$ of A with preimage $\mathfrak{m} = (\{A_{\nu\mu} - a_{\nu\mu}\})$ in R is given, then $\mathfrak{M}$ corresponds to a singularity of the curve C_α. Moreover

$$A_{\mathfrak{M}}/\mathfrak{m}A_{\mathfrak{M}} = (K[C_\alpha]/J)_{\overline{\mathfrak{M}}}$$

with the Jacobian ideal J of $K[C_\alpha]$ and the image $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ in $K[C_\alpha]/J$.

Proposition 1.1. *The singular curves of type p, q are the closed points of the irreducible hypersurface $\mathcal{H} \subset \mathbb{A}^n(K)$. The closed points of $\mathbb{A}^n(K)$ outside of $\mathcal{H}$ are in one-to-one correspondence with the smooth curves of type p, q. Their Weierstraß semigroup is H_{pq}.*

The last statement of the proposition follows from the fact that the Weierstraß semigroup of a smooth curve of type p, q has genus $g = d$, hence no gaps of H_{pq} have to be closed in it.

An example of a smooth curve of type p, q is given by the equation $Y^p - X^q + a_{00} = 0$ $(a_{00} \neq 0)$.

As an R-module A can be written

$$A = N / \sum_{\alpha < q-1, \beta < p-1} R \cdot \xi^\alpha \eta^\beta F(\xi, \eta)$$

and the relations

$$\xi^{q-1} = \frac{1}{q} \sum \nu A_{\nu\mu} \xi^{\nu-1} \eta^\mu, \ \eta^{p-1} = -\frac{1}{p} \sum \mu A_{\nu\mu} \xi^\nu \eta^{\mu-1} \tag{1}$$

allow with the usual reduction process to write

$$\xi^\alpha \eta^\beta F(\xi, \eta) = \sum_{\nu < q-1, \mu < p-1} r_{\nu\mu}^{\alpha\beta} \cdot \xi^\nu \eta^\mu \qquad (r_{\nu\mu}^{\alpha\beta} \in R).$$

The $c \times c$-matrix $M := \left(r_{\nu\mu}^{\alpha\beta}\right)$ represents the multiplication by $F(\xi, \eta)$ in N, and since $F(\xi, \eta)$ is not a zero-divisor in N we have an exact sequence of R-modules

$$0 \to R^c \xrightarrow{M} R^c \to A \to 0. \tag{2}$$

M is a relation matrix of the R-module A with respect to the basis B of N. For $0 \le l \le c$ the $(c-l)$-minors of M generate the l-th Fitting ideal $F_l(A/R)$ of the R-module A. In particular $F_0(A/R) = (\Delta)$ with $\Delta := \det(M)$, the norm of the multiplication map by $F(\xi, \eta)$. Here $\Delta \neq 0$, the map given by M being injective. We have

$$(0) \neq F_0(A/R) \subset F_1(A/R) \subset \cdots \subset F_c(A/R)$$

By [Ku1], D.14

$$\mathrm{Ann}_R(A)^c \subset F_0(A/R) = (\Delta) \subset \mathrm{Ann}_R(A) \tag{3}$$

and therefore $\mathrm{Rad}(\mathrm{Ann}_R(A)) = \mathrm{Rad}(\Delta)$. As A is an R-algebra $\mathrm{Ann}_R(A)$ is the kernel $\mathfrak{q}$ of the structure homomorphism $R \to A$, hence $\mathrm{Rad}(\Delta)$ is also a prime ideal. It follows that $\Delta = a\Delta_0^r$ with an irreducible polynomial Δ_0 of R which generates $\mathfrak{q}$, an $a \in K \setminus \{0\}$ and an $r \in \mathbb{N}$, hence $R' = R/\mathfrak{q} = R/(\Delta_0)$ and the hypersurface $\mathcal{H}$ is given by the equation $\Delta_0 = 0$.

For $\mathfrak{p} \in \mathrm{Spec}(R)$ and $l \in \{0, \ldots, c\}$ we have the following formula for the minimal number of generators of the $R_{\mathfrak{p}}$-module $A_{\mathfrak{p}}$

$$\mu_{\mathfrak{p}}(A) = \min\{l \mid F_l(A_{\mathfrak{p}}/R_{\mathfrak{p}}) = R_{\mathfrak{p}}\} \tag{4}$$

([Ku1], D.8).

Let $\mathfrak{m} = (\{A_{\nu\mu} - a_{\nu\mu}\})$ be a maximal ideal of R corresponding to the polynomial $\bar{F} := F(\alpha, X, Y) \in K[X, Y]$ and $l \in \{0, \ldots, c\}$. Then by (4) $F_l(A/R)_{\mathfrak{m}} = F_l(A_{\mathfrak{m}}/R_{\mathfrak{m}}) = R_{\mathfrak{m}}$ if and only if the $R_{\mathfrak{m}}$-module $A_{\mathfrak{m}}$ has a minimal number of generators $\le l$, that is, if and only if

$$\dim_K(K[X, Y]/(\bar{F}, \bar{F}_X, \bar{F}_Y)) \le l. \tag{5}$$

If $\mathfrak{M} \in \mathrm{Max}(K[X, Y])$ corresponds to a node of C_α, then

$$\dim_K((K[X, Y]/(\bar{F}, \bar{F}_X, \bar{F}_Y))_{\mathfrak{M}}) = 1.$$

If C_α is a nodal curve, then $\dim_K(K[X, Y]/(\bar{F}, \bar{F}_X, \bar{F}_Y))$ is the number of its nodes and (4) implies

Lemma 1.2. *If C_α has at most l nodes and no other singularities, then $\mathfrak{m}$ is contained in the open set $\mathrm{Max}(R) \setminus V(F_l(A/R))$ of $\mathrm{Max}(R)$. Conversely, if C_α has l distinct nodes and $\mathfrak{m} \in \mathrm{Max}(R) \setminus V(F_l(A/R))$, then C_α is a nodal curve with exactly l nodes.*

For the module of differentials of A/R we have

$$\Omega^1_{A/R} = AdX \oplus AdY / \langle F_{XX}(x, y)dX + F_{XY}(x, y)dY, F_{YX}(x, y)dX + F_{YY}(x, y)dY \rangle$$

with the residue classes x, y of X, Y in A. Since the variables A_{00}, A_{10}, A_{01} have disappeared in the second derivatives the Hesse determinant $\mathrm{Hess}_F(X, Y)$ of F is a non-zero polynomial in A. Take $\mathfrak{M} \in \mathrm{Max}(A)$ with preimage $\mathfrak{m}$ in R corresponding to a point in $\mathcal{H}$. Nodes are the singularties with non-vanishing Hesse determinant, hence $\mathfrak{M}$ corresponds to a node of the curve given by $\mathfrak{m}$ if and only if $\mathfrak{M} \in \mathrm{Max}(A) \setminus V(\mathrm{Hess}_F)$. This is equivalent to each of the following conditions

(i) $\mathrm{Hess}_F(X,Y)$ is a unit in $A_{\mathfrak{M}}$.

(ii) $\Omega^1_{A_{\mathfrak{M}}/R} = 0$.

(iii) $\mathfrak{M}$ is unramified over R ([Ku1], 6.10).

From (ii) and (iii) we conclude

Proposition 1.3. *The nodal curves of type p, q having at least one node correspond bijectively to the maximal ideals $\mathfrak{m} \in V(\Delta_0) = \mathcal{H}$ with $\mathfrak{m} \notin V(Ann_R(\Omega^1_{A/R}))$ or equivalently with A/R being unramified at $\mathfrak{m}$.*

We denote this open set of the hypersurface $\mathcal{H}$ by $\mathcal{H}_{pq}$. The additional assumption that $\mathfrak{m} \notin V(F_l(A/R))$ defines for each $l = 1, \dots, d$ an open subset U_l of $\mathcal{H}_{pq}$ whose closed points correspond to the nodal curves of type p, q having at most l nodes. Set $U_0 := \emptyset$. We have

$$\mathcal{H}_{pq} = \bigcup_{l=1}^{d} \mathcal{H}^l_{pq}$$

with the locally closed subset $\mathcal{H}^l_{pq} := U_l \setminus U_{l-1}$ whose closed points correspond to the curves having exactly l nodes. The Weierstraß semigroups of the curves in $\mathcal{H}^l_{pq}$ are certain semigroups H with $p, q \in H$ having genus $g = d - l$. By [KKW], Theorem 6.4 every Weierstraß semigroup H of genus g is the Weierstraß semigroup of an element of $\mathcal{H}^{d-g}_{pq}$ for suitably chosen p, q.

The curve $C : (Y-b)^p - (X-a)^q + c(X-a)(Y-b) = 0$ $(a, b, c \in K, c \neq 0))$ has only one singularity at (a, b), and it is a node. Therefore $\mathcal{H}^1_{pq} \neq \emptyset$. The associated normed curve of the Lissajous curve of type p, q ([KKW], Example 2.4) has the maximal possible number d of nodes, hence $\mathcal{H}^d_{pq} \neq \emptyset$.

For a domain B let $Q(B)$ denote its quotient field.

Proposition 1.4. *We have $Q(R') = Q(A)$. Hence $R' \to A$ induces a finite birational morphism $\mathbb{A}^{n-1}(K) \to \mathcal{H}$, and the hypersurface $\mathcal{H}$ is rational.*

Proof. Let $\mathfrak{p}$ be the kernel of $R \to A$. The inclusion $R' \to A$ induces an injection $Q(R') \to A_{\mathfrak{p}}$. Since A is a domain and integral over R' we have $A_{\mathfrak{p}} = Q(A)$. Moreover $F_1(A_{\mathfrak{p}}/R_{\mathfrak{p}}) = R_{\mathfrak{p}}$ since $F_1(A_{\mathfrak{m}}/R_{\mathfrak{m}}) = R_{\mathfrak{m}}$ with the $\mathfrak{m}$ belonging to the curve C above, as Fitting ideals are compatible with localization. Hence by (4) $A_{\mathfrak{p}}$ is generated over $R_{\mathfrak{p}}$ by one element, i.e. $Q(A) = Q(R')$. $\square$

For the maximal ideals $\mathfrak{m} \in \mathcal{H}^l_{pq}$ all $\mathfrak{M} \in \mathrm{Max}(A)$ lying over $\mathfrak{m}$ are unramified over R. Let $\mathfrak{m} = (\{A_{\nu\mu} - a_{\nu\mu}\}_{\nu p + \mu q < pq}) \in \mathcal{H}^l_{pq}$ with $\alpha := (\{a_{\nu\mu}\}) \in K^n$ be given, and let $\mathfrak{M} \in \mathrm{Max}(A)$ correspond to a node of the curve C_α. Set $T_{\nu\mu} := A_{\nu\mu} - a_{\nu\mu}$ for short. The canonical homomorphism $R_{\mathfrak{m}} \to A_{\mathfrak{M}}$ induces a local homomorphism $\varphi : \widehat{R_{\mathfrak{m}}} \to \widehat{A_{\mathfrak{M}}}$ of the completions which is surjective since $A_{\mathfrak{M}}$ is unramified over $R_{\mathfrak{m}}$. Here $\widehat{R_{\mathfrak{m}}} = K[[\{T_{\nu\mu}\}]]$ and $\widehat{A_{\mathfrak{M}}}$ are regular local rings of dimension n resp. $n-1$. Therefore $\ker(\varphi)$ is generated by a power series $\Delta_{\mathfrak{M}}$ of order 1, an irreducible factor of Δ_0 considered as a power series in the $T_{\nu\mu}$.

Thus $\mathfrak{M}$ defines a smooth analytic branch $\mathrm{Spec}(\widehat{R_\mathfrak{m}}/(\Delta_\mathfrak{M}))$ of $\mathcal{H}$ near the point α. Different nodes of C_α define different branches as the power series Δ_0 cannot have multiple factors, Δ_0 being an irreducible polynomial. The local ring $R_\mathfrak{m}/(\Delta_0)$ is regular if and only if C_α has only one node. Thus $\mathcal{H}^1_{pq}$ is the set of regular points of $\mathcal{H}_{pq}$.

Let $\widehat{A_\mathfrak{m}}$ be the completion of $A_\mathfrak{m} := R_\mathfrak{m} \otimes A$ as an $R_\mathfrak{m}$-module. Then

$$\widehat{A_\mathfrak{m}} = \widehat{A_{\mathfrak{M}_1}} \times \cdots \times \widehat{A_{\mathfrak{M}_l}} = \widehat{R_\mathfrak{m}}/(\Delta_{\mathfrak{M}_1}) \times \cdots \times \widehat{R_\mathfrak{m}}/(\Delta_{\mathfrak{M}_l}) \tag{6}$$

by the Chinese Remainder Theorem. Since the Fitting ideals are compatible with localization and completion we obtain that

$$F_0(\widehat{A_\mathfrak{m}}/\widehat{R_\mathfrak{m}}) = \widehat{R_\mathfrak{m}} \cdot F_0(A/R) = \widehat{R_\mathfrak{m}} \cdot \Delta = \widehat{R_\mathfrak{m}} \cdot (\prod_{i=1}^{l} \Delta_{\mathfrak{M}_i}) = \widehat{R_\mathfrak{m}} \cdot \Delta_0.$$

Remember that $\Delta = a\Delta_0^r$ with $a \in K \setminus \{0\}$ and $r \geq 1$. Since we know that nodal curves of type p, q with at least one node exist for every p, q the above consideration implies that $r = 1$ and we have proved the irreducibility of Δ, the polynomial generating $F_0(A/R)$. Thus the hypersurface $\mathcal{H}$ is defined by $F_0(A/R) = (\Delta)$.

We determine the leading form of $\Delta_\mathfrak{M}$ which defines the tangent hyperplane of the branch $\Delta_\mathfrak{M} = 0$. As $\Omega^1_{A_\mathfrak{M}/R} = 0$ we see that $\Omega^1_{A_\mathfrak{M}/K}$ is generated by the differentials $dT_{\nu\mu}$ ($\nu p + \mu q < pq$). Moreover we have the relation

$$\sum_{\nu p + \mu q < pq} x^\nu y^\mu dT_{\nu\mu} = 0 \tag{7}$$

coming from $dF = 0$. Therefore $\{dT_{\nu\mu}\}_{(\nu,\mu)\neq(0,0)}$ is a basis of $\Omega^1_{A_\mathfrak{M}/K}$, and we obtain

$$\widehat{\Omega^1_{A_\mathfrak{M}/K}} = \bigoplus_{(\nu,\mu)\neq(0,0)} \widehat{A_\mathfrak{M}} \cdot dT_{\nu\mu}$$

In $\widehat{\Omega^1_{A_\mathfrak{M}/K}}$ there is the relation $d\Delta_\mathfrak{M} = \sum_{\nu p + \mu q < pq} \partial\Delta_\mathfrak{M}/\partial T_{\nu\mu} \cdot dT_{\nu\mu}$, and by (7)

$$\sum_{(\nu,\mu)\neq(0,0)} \big(\partial\Delta_\mathfrak{M}/\partial T_{\nu\mu} - x^\nu y^\mu \cdot \partial\Delta_\mathfrak{M}/\partial T_{00}\big) dT_{\nu\mu} = 0$$

which implies in $\widehat{A_\mathfrak{M}}$ the relations

$$\partial\Delta_\mathfrak{M}/\partial T_{\nu\mu} = x^\nu y^\mu \cdot \partial\Delta_\mathfrak{M}/\partial T_{00}$$

for all ν, μ. Since $\Delta_\mathfrak{M}$ has order 1, at least one of the partial derivatives must be a unit in $\widehat{A_\mathfrak{M}}$, hence so must be the partial with respect to T_{00}. Let $(\xi, \eta) \in K^2$ be the node corresponding to $\mathfrak{M}$. Considering the above relations modulo $\mathfrak{M}\widehat{A_\mathfrak{M}}$ we find that

$$\partial\Delta_\mathfrak{M}/\partial T_{\nu\mu}\,|_{\,0} = \xi^\nu \eta^\mu \cdot \partial\Delta_\mathfrak{M}/\partial T_{00}\,|_{\,0}$$

where the last partial does not vanish, hence the leading form of $\Delta_{\mathfrak{M}}$ is

$$L_{\mathfrak{M}}\Delta_{\mathfrak{M}} = \partial\Delta_{\mathfrak{M}}/\partial T_{00}\,|_{\,0} \cdot \sum_{\nu p+\mu q<pq} \xi^\nu \eta^\mu T_{\nu\mu}. \tag{8}$$

Collecting everything we obtain

Proposition 1.5. *At the closed points of $\mathcal{H}^l_{pq}$ the hypersurface $\mathcal{H}$ has l regular branches with tangent hyperplanes given by (8).*

Let $(\xi_1,\eta_1),\dots,(\xi_l,\eta_l)$ be the nodes of C_α. We shall see in Lemma 2.1 that the matrix $(\xi_i^\nu\eta_i^\mu)_{\nu p+\mu q<pq, i=1,\dots,l}$ has rank l. Thus the $L_{\mathfrak{M}_i}\Delta_{\mathfrak{M}_i}$ $(i=1,\dots,l)$ are linearly independent over K and the $\Delta_{\mathfrak{M}_i}$ form part of a regular system of parameters of $\widehat{R_{\mathfrak{m}}}$.

For the defining polynomial Δ of $\mathcal{H}$ this means the following: If we expand Δ as a polynomial in the $T_{\nu\mu}=A_{\nu\mu}-a_{\nu\mu}$ its form of lowest degree is up to a constant factor the product of the l homogenous linear polynomials $L_{\mathfrak{M}_i}\Delta_{\mathfrak{M}_i}$ which moreover are linearly independent over K.

Formula (6) implies that

$$\widehat{R_{\mathfrak{m}}}\cdot F_k(A/R) = F_k(\widehat{A_{\mathfrak{m}}}/\widehat{R_{\mathfrak{m}}}) = (\{\Delta_{\mathfrak{M}_{i_1}}\cdot\dots\cdot\Delta_{\mathfrak{M}_{i_{l-k}}}\}_{i_1<\dots<i_{l-k}}).$$

Let $\mathfrak{p}_{i_1,\dots,i_k}$ be the prime ideal of $\widehat{R_{\mathfrak{m}}}$ generated by $\Delta_{\mathfrak{M}_{i_1}},\dots,\Delta_{\mathfrak{M}_{i_k}}$ where $i_1<\dots<i_k$. Then

$$\widehat{R_{\mathfrak{m}}}\cdot F_k(A_{\mathfrak{m}}/R_{\mathfrak{m}}) = \bigcap_{i_1<\dots<i_{k+1}} \mathfrak{p}_{i_1,\dots,i_{k+1}}\ (k=0,\dots,l-1) \tag{9}$$

in particular

$$\widehat{R_{\mathfrak{m}}}\cdot F_{l-1}(A/R) = F_{l-1}(\widehat{A_{\mathfrak{m}}}/\widehat{R_{\mathfrak{m}}}) = (\Delta_{\mathfrak{M}_1},\dots,\Delta_{\mathfrak{M}_l}) = \mathfrak{p}_{1,\dots,l}.$$

One can prove (9) by first showing it when the $\Delta_{\mathfrak{M}_i}$ are variables in a polynomial ring and by passing then to the completion. Thus the ideals $\widehat{R_{\mathfrak{m}}}\cdot F_k(A/R)$ $(k=0,\dots,l-1)$ are radical ideals of height $k+1$ in $\widehat{R_{\mathfrak{m}}}$, and so are the $F_k(A_{\mathfrak{m}}/R_{\mathfrak{m}})$ in $R_{\mathfrak{m}}$.

Theorem 1.6. *For any l with $1\le l\le d$ there is a nodal curve of type p,q with exactly l nodes, i.e. $\mathcal{H}^l_{pq}\neq\emptyset$.*

Proof. Let $\mathfrak{m}\in\operatorname{Max}(R)$ correspond to the curve C associated to the Lissajous curve of type p,q. There is a $g\in R$ such that $\mathfrak{m}\in D(g)$ and that the closed points in $D(g)\cap\mathcal{H}$ correspond to nodal curves. Then by the above the $F_k(A_g/R_g)$ $(k=0,\dots,d)$ form a strictly increasing sequence of radical ideals in R_g. Choose a maximal ideal $\mathfrak{n}\in D(g)$ such that

$$F_{l-1}(A_g/R_g)\subset\mathfrak{n}R_g,\ F_l(A_g/R_g)\not\subset\mathfrak{n}R_g.$$

Then the curve corresponding to $\mathfrak{n}$ has exactly l nodes. ∎

Examples 1.7. The Weierstraß semigroups of the curves in $\mathcal{H}_{pq}^l$ are numerical semigroups H with $p, q \in H$ and genus $d - l$. If $l \leq p/2$ all possible H of this kind do occur, see [KKW], Example 5.4. For $l = 1$ the semigroup H is obtained from H_{pq} by closing one gap $c - 1 - (ap + bq), (a, b \in \mathbb{N})$. We must have $a = b = 0$, otherwise more than one gap would be closed. Therefore $H = \langle p, q, c - 1 \rangle$ and any curve in $\mathcal{H}_{pq}^1$ has this Weierstraß semigroup. In $\mathcal{H}_{p,q}^2$ we have the Weierstraß semigroups $\langle p, q, c - 1 - p \rangle$ and $\langle p, q, c - 1 - q \rangle$. The curves in $\mathcal{H}_{pq}^d$ are the nodal curves of type p, q for which the normalization of its projective closure has genus 0. Their Weierstraß semigroup is $\mathbb{N}$.

The hypersurface $\mathcal{H}$ contains many lines.

Proposition 1.8. *Let $\alpha \neq \beta$ be closed points of $\mathcal{H}$ such that $\mathrm{Sing}(C_\alpha) \cap \mathrm{Sing}(C_\beta) \neq \emptyset$, and let L be the line through α and β. Then $L \subset \mathcal{H}$, and for almost all closed $\gamma \in L$ the curve C_γ has the singular set $\mathrm{Sing}(C_\alpha) \cap \mathrm{Sing}(C_\beta)$.*

Proof. Set $H := F(\beta, X, Y) - F(\alpha, X, Y)$ and $D := V(H)$. Then H and $F(\alpha, X, Y)$ are relatively prime and $\mathrm{Sing}(C_\alpha) \cap \mathrm{Sing}(D) = \mathrm{Sing}(C_\alpha) \cap \mathrm{Sing}(C_\beta)$. By [KKW], Proposition 3.1 the curve

$$F(\alpha, X, Y) + d \cdot H = F(\alpha + d(\beta - \alpha), X, Y) = 0$$

has for almost all $d \in K \setminus \{0\}$ the singular set $\mathrm{Sing}(C_\alpha) \cap \mathrm{Sing}(C_\beta) \neq \emptyset$. It follows that $L \subset \mathcal{H}$. $\square$

Corollary 1.9. *For any closed point $\alpha \in \mathcal{H}$ there is at least one line L with $\alpha \in L \subset \mathcal{H}$.*

Proof. Let (a, b) be a singularity of C_α, and let C_β be a nodal curve with (a, b) as its only node. It can be chosen such that $\alpha \neq \beta$. Then H contains by 1.8 the line through α and β. $\square$

Corollary 1.10. *Let $L \subset \mathcal{H}$ be a line through a closed point α where C_α is a nodal curve. Then for almost all closed points $\gamma \in L$ the curves C_γ have the same Weierstraß semigroup.*

Proof. Since $\alpha \in \mathcal{H}_{pq}$ and this set is open in $\mathcal{H}$ almost all C_γ with $\gamma \in L$ are nodal curves having by 1.8 the same set of nodes. By [KKW], Corollary 4.3 they also have the same Weierstraß semigroup. $\square$

2 Which numerical semigroups are Weierstraßsemigroups?

Let H be a numerical semigroup of genus g and let $p < q$ be relatively prime numbers from H. The semigroup H_{pq} has d gaps $\gamma_1 < \cdots < \gamma_d$ which can be written

$$\gamma_i = (p-1)(q-1) - 1 - (a_i p + b_i q) \ (i = 1, \ldots, d)$$

with a unique $(a_i, b_i) \in \mathbb{N}^2$. Of these gaps $l := d - g$ are closed in H. We want to decide whether a nodal curve C of type p, q with l nodes exists such that H is the Weierstraß semigroup of C.

Let $\gamma_{j_1} < \cdots < \gamma_{j_l}$ be the gaps of H_{pq} which are closed in H. Further let $\mathcal{A}_H(X_1, Y_1, \ldots, X_l, Y_l)$ be the matrix

$$\begin{pmatrix} X_1^{a_{j_1}} Y_1^{b_{j_1}} & \ldots & X_1^{a_{j_l}} Y_1^{b_{j_l}} \\ \vdots & \ldots & \vdots \\ X_l^{a_{j_1}} Y_l^{b_{j_1}} & \ldots & X_l^{a_{j_l}} Y_l^{b_{j_l}} \end{pmatrix}$$

and $D_H(X_1, Y_1, \ldots, X_l, Y_l) := \det(\mathcal{A}_H(X_1, Y_1, \ldots, X_l, Y_l))$ its determinant.

Lemma 2.1. *If H is the Weierstraß semigroup of a nodal curve $C : F = 0$ of type p, q with the nodes $(\xi_1, \eta_1), \ldots, (\xi_l, \eta_l)$, then $D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0$.*

Proof. If $D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) = 0$, then there exists a non-zero $\lambda = (\lambda_1, \ldots, \lambda_l) \in K^l$ such that $\mathcal{A}_H \cdot \lambda^t = 0$. Assume that $\lambda_1 = \cdots = \lambda_{i-1} = 0,\ \lambda_i \neq 0$. Let x, y denote the images of X, Y in the function field $K(C)$ of C and P the place at infinity of C. The function

$$\Phi(x, y) := \lambda_i x^{a_{j_i}} y^{b_{j_i}} + \cdots + \lambda_l x^{a_{j_l}} y^{b_{j_l}} \in K[C]$$

satisfies $\Phi(\xi_i, \eta_i) = 0$ $(i = 1, \ldots, l)$. If follows from [KKW], Proposition 4.2 that $\mathrm{ord}_P(\frac{\Phi(x,y)}{F_Y(x,y)} dx) + 1 = \gamma_{j_i}$ is a gap of H, contradicting the fact that γ_{j_i} was a gap of H_{pq} closed in H. □

With the generic Weierstraß polynomial $F(\{A_{\nu\mu}\}, X, Y) \in R[X, Y]$ of type p, q and l with $1 \leq l \leq d$ set

$$T := R[X_1, Y_1, \ldots, X_l, Y_l]/(\{F(X_i, Y_i), F_X(X_i, Y_i), F_Y(X_i, Y_i)\}_{i=1,\ldots,l}).$$

Let $C_L : F(\{a^L_{\nu\mu}\}, X, Y) = 0$ be the normed curve associated to the Lissajous curve of type p, q, and let (ξ_i, η_i) $(i = 1, \ldots, d)$ be its nodes. C_L has the Weierstraß semigroup $\mathbb{N}$. By Lemma 2.1 we have $D_{\mathbb{N}}(\xi_1, \eta_1 \ldots, \xi_d, \eta_d) \neq 0$. Therefore the columns of this determinant corresponding to the gaps $\gamma_{j_1}, \ldots, \gamma_{j_l}$ are linearly independent over K, and there are nodes $(\xi_1, \eta_1) \ldots, (\xi_l, \eta_l)$ (say) such that $D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0$ too.

Let δ be the image of $D_H(X_1, Y_1, \ldots, X_l, Y_l)$ and t that of $\prod_{i=1}^{l} \mathrm{Hess}_F(X_i, Y_i)$ in T. Then $t \cdot \delta$ is not contained in the maximal ideal corresponding to the point $(\{a^L_{\nu\mu}\}, \xi_1, \eta_1, \ldots, \xi_l, \eta_l)$ and hence $t \cdot \delta$ is not nilpotent. Therefore $S_H := T_{t \cdot \delta}$ is not the zero-ring. Now the elements of $\mathrm{Max}(S_H)$ correspond bijectively to the $(\beta, \xi_1, \eta_1, \ldots, \xi_l, \eta_l)$ where the (ξ_i, η_i) are nodes of the curve C_β and have the additional property that $D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0$. In particular the nodes are distinct.

Let h be the set of the $(a_{j_i}, b_{j_i}) \in \mathbb{N}^2$ $(i = 1, \ldots, l)$ corresponding to the gaps of H_{pq} which are closed in H. Let x_i, y_i be the images of the X_i, Y_i in S_H and denote the images of the $A_{\nu\mu}$ also by $A_{\nu\mu}$ $(\nu p + \mu q < pq)$.

Lemma 2.2. *We have $\Omega^1_{S_H/K} = \bigoplus_{(\nu,\mu) \notin h} S_H dA_{\nu\mu}$. In particular S_H is a regular K-algebra, equidimensional of dimension $n - l$. Further S_H is unramified over $K[\{A_{\nu\mu}\}_{(\nu,\mu) \notin h}]$.*

Proof. The module of differentials has the presentation

$$\Omega^1_{S_H/K} = \bigoplus_{\nu p+\mu q<pq} S_H dA_{\nu\mu} \oplus \bigoplus_{i=1}^{l} S_H dX_i \oplus S_H dY_i / U$$

where U is generated by

$$\sum_{\nu p+\nu q<pq} x_i^\nu y_i^\mu dA_{\nu\mu},$$

$$\sum_{\nu p+\nu q<pq} \nu x_i^{\nu-1} y_i^\mu dA_{\nu\mu} + F_{XX}(x_i,y_i)dX_i + F_{XY}(x_i,y_i)dY_i$$

and

$$\sum_{\nu p+\nu q<pq} \mu x_i^{\nu} y_i^{\mu-1} dA_{\nu\mu} + F_{YX}(x_i,y_i)dX_i + F_{YY}(x_i,y_i)dY_i$$

$(i=1,\dots,l)$. Since $\mathrm{Hess}_F(x_i,y_i)$ $(i=1,\dots,l)$ and $D_H(x_1,y_1,\dots,x_l,y_l)$ are units in S_H the statement about $\Omega^1_{S_H/K}$ follows, and the remaining assertions are clear by the differential criterion of regularity ([Ku1],7.2). $\square$

Now let $U^l_{pq}(H) := \mathrm{Spec}(S_H) \setminus V(F_l(A/R)S_H)$. By Lemma 1.2 the closed points $(\alpha,\xi_1,\eta_1,\dots,\xi_l,\eta_l)$ of the scheme $U^l_{pq}(H)$ are those for which the curve C_α has no singularities but the nodes (ξ_i,η_i) which satisfy $D_H(\xi_1,\eta_1,\dots,\xi_l,\eta_l) \neq 0$. These C_α have a Weierstraß semigroup which is obtained from H_{pq} by closing l of its gaps, but may be different from H.

It will be shown in Proposition 3.1 that the scheme $U^l_{pq}(H)$ is not empty. In order to decide whether H is the Weierstraß semigroup of a nodal curve of type p,q we need a further consideration which is inspired by [Ha], IV.4.

Let $\gamma_{i_1} < \dots < \gamma_{i_g}$ be the gaps of H, $\gamma_{i_k} = c-1-(a_{i_k}p + b_{i_k}q)$. Then

$$\{\gamma_{i_1},\dots,\gamma_{i_g}\} \cup \{\gamma_{j_1}\dots,\gamma_{j_l}\}$$

is the set of all gaps of H_{pq}. In H_{pq} there are $d-i_k$ gaps $>\gamma_{i_k}$, and H has $g-k$ gaps $>\gamma_{i_k}$. Hence there are $(d-i_k)-(g-k) = l-(i_k-k)$ gaps of H_{pq} which are $>\gamma_{i_k}$ and are closed in H. Therefore $\gamma_{j_m} > \gamma_{i_k}$ if and only if $m > i_k - k$.

Let s_k be the column

$$\begin{pmatrix} X_1^{a_{i_k}} Y_1^{b_{i_k}} \\ \vdots \\ \vdots \\ X_l^{a_{i_k}} Y_l^{b_{i_k}} \end{pmatrix} \quad (k=1,\dots,g)$$

and $D_k^m(X_1,Y_1,\dots,X_l,Y_l)$ for $m \in \{1,\dots,i_k-k\}$ the determinant of the matrix which is obtained from $\mathcal{A}_H$ by replacing its m-th column by s_k. These are $\sum_{k=1}^g (i_k-k) = \sum_{k=1}^g i_k - \binom{g+1}{2}$ determinants. Let J be the ideal generated by their images in S_H. If the semigroup H is obtained from H_{pq} by closing its l greatest gaps, then no D_k^m are present, and we set $J=(0)$. Let $V_{pq}(H) := U^l_{pq}(H) \cap V(J)$.

Theorem 2.3. *The closed points of $V_{pq}(H)$ correspond to the nodal curves of type p, q having the Weierstraß semigroup H, i.e. H is the Weierstraß semigroup of such a curve if and only if $V_{pq}(H) \neq \emptyset$.*

Proof. a) Let $Q := (\alpha, \xi_1, \eta_1, \dots, \xi_l, \eta_l) \in V_{pq}(H)$. Since $Q \in U^l_{pq}(H)$ the curve C_α is a nodal curve with the nodes $(\xi_1, \eta_1), \dots, (\xi_l, \eta_l)$ and $D_H(\xi_1, \eta_1, \dots, \xi_l, \eta_l) \neq 0$. Moreover

$$D^m_k(\xi_1, \eta_1, \dots, \xi_l, \eta_l) = 0 \text{ for } k = 1, \dots, g \text{ and } m = 1, \dots, i_k - k. \tag{1}$$

Further for any $k \in \{1, \dots, g\}$ the linear system of equations

$$\mathcal{A}_H(\xi_1, \eta_1, \dots, \xi_l, \eta_l) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_l \end{pmatrix} = -s_k(\xi_1, \eta_1, \dots, \xi_l, \eta_l)$$

has a unique solution. By Cramer's rule (1) implies that $\lambda_1 = \dots = \lambda_{i_k - k} = 0$. Let x, y denote the images of X, Y in the function field of C_α. The polynomial

$$\Phi_k(X, Y) := X^{a_{i_k}} Y^{b_{i_k}} + \sum_{m > i_k - k} \lambda_m X^{a_{j_m}} Y^{b_{j_m}}$$

vanishes at the nodes (ξ_i, η_i) $(i = 1, \dots, l)$, and since $\gamma_{j_m} > \gamma_{i_k}$ for $m > i_k - k$ the differential $\omega_k := \frac{\Phi_k(x,y)}{F_Y(x,y)} dx$ has order $\mathrm{ord}_P(\omega_k) = \gamma_{i_k} - 1$ at the place at infinity of C_α. By [KKW], Proposition 4.2 $\gamma_{i_1}, \dots, \gamma_{i_g}$ are gaps of the Weierstraß semigroup of C_α, i.e. H is this semigroup.

b) Let H be the Weierstraß semigroup of a nodal curve $C_\alpha : F(\alpha, X, Y) = 0$ of type p, q with l distinct nodes $(\xi_1, \eta_1), \dots, (\xi_l, \eta_l)$. We show that $Q := (\alpha, \xi_1, \eta_1, \dots, \xi_l, \eta_l) \in V_{pq}(H)$. By the discussion above we know already that $Q \in U^l_{pq}(H)$.

Let Ω_∞ be the vector space of differentials with non-negative order at the place P at infinity of C_α. According to [KKW], Lemma 4.1 we can choose a basis $\{\omega_1, \dots, \omega_l\}$ of the vector space Ω of holomorphic differentials on $\mathcal{R}$ such that $\omega_k = \frac{\Phi_k(x,y)}{F_Y(x,y)} dx$ with

$$\Phi_k(x, y) = x^{a_{i_k}} y^{b_{i_k}} + \lambda_{i_k+1} x^{a_{i_k+1}} y^{b_{i_k+1}} + \dots + \lambda_d x^{a_d} y^{b_d} \ (k = 1, \dots, g)$$

and $\mathrm{ord}_P(\omega_k) + 1 = \gamma_{i_k}$. By elementary transformations we attain that

$$\Phi_k(x, y) = x^{a_{i_k}} y^{b_{i_k}} + \tilde{\lambda}_{r,k} x^{a_{j_r}} y^{b_{j_r}} + \dots + \tilde{\lambda}_{l,k} x^{a_{j_l}} y^{b_{j_l}} \ (k = 1, \dots, g),$$

with certain $\tilde{\lambda}_{i,k} \in K$ where $r = i_k - k + 1$. Since $\Phi_k(\xi_i, \eta_i) = 0$ $(i = 1, \dots, l)$ and $\tilde{\lambda}_{k,m} = 0$ $(m = 1, \dots, i_k - k)$ Cramer's rule implies that $D^m_k(\xi_1, \eta_1, \dots, \xi_l, \eta_l) = 0$ for $k = 1, \dots, g$ and $m = 1, \dots, i_k - k$. Hence $Q \in V(J) \cap U^l_{pq}(H) = V_{pq}(H)$. $\square$

Theorem 2.3 and [KKW], Theorem 6.4 imply

Corollary 2.4. *Let p be greater than the elements of the minimal system of generators of H. Then H is a Weierstraß semigroup if and only if $V_{pq}(H) \neq \emptyset$.*

The closed points of $V_{pq}(H)$ are the $(\{a_{\nu\mu}\}_{\nu p+\mu q<pq}, \xi_1, \eta_1, \ldots, \xi_l, \eta_l) \in K^{n+2l}$ which are zeros of the polynomials

$$F(X_i, Y_i),\ F_X(X_i, Y_i),\ F_Y(X_i, Y_i)\ (i = 1, \ldots, l) \tag{2}$$

and of

$$D_k^m(X_1, Y_1, \ldots, X_l, Y_l)\ (k = 1, \ldots, g, m = 1, \ldots, i_k - k) \tag{3}$$

and not zeros of the polynomials $D_H(X_1, Y_1, \ldots, X_l, Y_l), \mathrm{Hess}_F(X_i, Y_i)$ $(i = 1, \ldots, l)$ and of at least one of the $N := \binom{c}{l}^2$ $(c - l)$-minors h_t $(t = 1, \ldots, N)$ of the matrix $M = \left(r_{\nu\mu}^{\alpha\beta}\right)$ defined in Section 1. Let $\mathfrak{a}$ be the ideal in $R[X_1, Y_1, \ldots, X_l, Y_l]$ generated by the polynomials (2) and (3). By Theorem 2.3 and Hilbert's Nullstellensatz H is the Weierstraß semigroup of a nodal curve of type p, q if and only if there exists $t \in \{1, \ldots, N\}$ such that

$$h_t \cdot \prod_{i=1}^{l} \mathrm{Hess}_F(X_i, Y_i) \cdot D_H(X_1, Y_1, \ldots, X_l, Y_l) \notin \mathrm{Rad}(\mathfrak{a}). \tag{4}$$

One can try to decide this by the radical membership test (see e.g. [Kr-R], page 219). However the number N of necessary tests increases rapidly with p and q, and so do the degrees of the involved polynomials. A sufficient condition is that (4) holds for a $(c-1)$-minor h_t of the matrix M which requires c^2 tests in the worst case, but with no guarantee of a success.

The polynomials in (2),(3) and (4) all belong to $\mathbb{Q}[\{A_{\nu\mu}\}, X_1, Y_1, \ldots, X_l, Y_l]$. Therefore (4) holds true if and only if it holds true for $K = \overline{\mathbb{Q}}$, the field of algebraic numbers. In other words, the property of H to be a Weierstraß semigroup is independent of the choice of the base field. For example we can test it for $K = \mathbb{C}$.

The projection $\mathbb{A}^{n+2l}(K) \to \mathbb{A}^n(K)$ $((\alpha, \xi_1, \eta_1, \ldots, \xi_l, \eta_l) \mapsto \alpha)$ maps the locally closed set $V_{pq}(H)$ onto a *constructible* set $V_{pq}^H \subset \mathcal{H}_{pq}^l$ whose closed points correspond bijectively to the nodal curves of type p, q with the Weierstraß semigroup H. We have

$$\mathcal{H}_{pq}^l = \bigcup_H V_{pq}^H$$

where H runs over the numerical semigroups containing p and q with $d - l$ gaps.

3 Simplification of nodal curves and a class of Weierstraß semigroups

Let $1 < p < q$ be relatively prime integers and $d = {}^1\!/_2(p-1)(q-1)$. In Theorem 1.6 we have seen that for any $l \in \{1, \ldots, d\}$ there is a nodal curve of type p, q with exactly l nodes. The following proposition gives a more precise statement and a different proof.

Proposition 3.1. *Let H be a numerical semigroup which is obtained from H_{pq} by closing l of its gaps. Then $U_{pq}^l(H) \neq \emptyset$.*

As an immediate consequence we get

Theorem 3.2. *Let H be the numerical semigroup which is obtained from H_{pq} by closing its l greatest gaps. Then H is a Weierstraß semigroup.*

In fact, for H as in 3.2 no determinants D_k^m occur. Therefore $V_{pq}(H) = U_{pq}^l(H)$ which is not empty by 3.1, and Theorem 2.3 implies that H is the Weierstraß semigroup of a nodal curve of type p, q. □

In order to prove 3.1 we need some preparations. Since $U_{pq}^l(H)$ is defined over $\overline{\mathbb{Q}}$ we may assume that $K = \mathbb{C}$. Let $R := \mathbb{C}[\{A_{\nu\mu}\}]$ and $F \in R[X, Y]$ the generic Weierstraß polynomial of type p, q. We have $\mathrm{Spec}(R) = \mathbb{A}^n(\mathbb{C})$ with $n = {}^1\!/_2(p+1)(q+1) - 1$. In $\mathrm{Spec}(R[X, Y]) = \mathbb{A}^n(\mathbb{C}) \times \mathbb{A}^2(\mathbb{C})$ we consider the smooth subschemes $V(F, F_X, F_Y) \cong \mathbb{A}^{n-1}(\mathbb{C})$ and $V(F_X, F_Y) \cong \mathbb{A}^n(\mathbb{C})$. Let $R' = R/(\Delta)$ be the image of R in $R[X, Y]/(F, F_X, F_Y)$ and

$$\mathcal{H}_{pq}^l \subset \mathcal{H}_{pq} \subset \mathcal{H} = \mathrm{Spec}(R') \subset \mathrm{Spec}(R) = \mathbb{A}^n(\mathbb{C})$$

as in Section 1. Further let $\pi : \mathbb{A}^n(\mathbb{C}) \times \mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^n(\mathbb{C})$ be the projection onto the first factor. Its restriction $\pi_0 : V(F, F_X, F_Y) \to \mathbb{A}^n(\mathbb{C})$ to $V(F, F_X, F_Y)$ is finite and has image $\mathcal{H}$. For a closed point $\alpha \in \mathcal{H}_{pq}^l$ the corresponding curve C_α has l nodes $(x_1, y_1), \ldots, (x_l, y_l)$ and no other singularities.

We endow $\mathbb{C}^m$ $(m > 0)$ with its standard norm $|| \; ||$ and standard topology. For $P \in \mathbb{C}^m$ and $\epsilon > 0$ let $U_\epsilon(P) := \{Q \in \mathbb{C}^m \mid \; ||Q - P|| < \epsilon\}$ denote the ϵ-neighborhood of P.

The proof of the following proposition is inspired by arguments of Benedetti-Risler [BR], Lemma 5.5.9 and Pecker [P] in real algebraic geometry.

Proposition 3.3 (Simplification of nodal curves)**.** *Let $P_{i_1}, \ldots, P_{i_\lambda}$ be distinct nodes of C_α $(1 \leq \lambda \leq l)$. Given $\epsilon > 0$ and $\delta > 0$ there exists $\beta \in U_\epsilon(\alpha)$ such that the curve $C_\beta : F(\beta, X, Y) = 0$ has λ distinct nodes $Q_1, \ldots, Q_\lambda$ and no other singularities where $Q_k \in U_\delta(P_{i_k})$ for $k = 1, \ldots, \lambda$.*

We obtain Proposition 3.1 by applying 3.3 to the normed curve C_α associated to the Lissajous curve of type p, q. Let (x_i, y_i) $(i = 1, \ldots, d)$ be the nodes of C_α and $\gamma_i = (p-1)(q-1) - 1 - (a_i p + b_i q)$ $(i = 1, \ldots, d)$ the gaps of H_{pq}. Then the determinant

$$D_{\mathbb{N}}(x_1, y_1, \ldots, x_d, y_d) = \det\Big(\Big(x_i^{a_j} y_i^{b_j}\Big)_{i,j=1,\ldots,d}\Big)$$

does not vanish by Lemma 2.1. Let γ_{j_k} $(k = 1, \ldots, l)$ be the gaps of H_{pq} which are closed in H. Consider the columns of $\left(x_i^{a_j} y_i^{b_j}\right)$ corresponding to the (a_{j_k}, b_{j_k}) $(k = 1, \ldots, l)$. Since they are linearly independent there exist nodes $P_{i_k} := (x_{i_k}, y_{i_k})$ of the curve C_α such that

$$D_H(x_{i_1}, y_{i_1}, \ldots, x_{i_l}, y_{i_l}) \neq 0.$$

By Proposition 3.3 there is a nodal curve $C_\beta : F(\beta, X, Y) = 0$ with exactly l nodes $Q_k = (\xi_k, \eta_k)$ $(k = 1, \ldots, l)$ which are arbitrarily close to the P_{i_k}. Then for a suitable β also $D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0$, and it follows that $(\beta, \xi_1, \eta_1, \ldots, \xi_l, \eta_l) \in U_{pq}^l(H)$. □

Proposition 3.3. In the following we consider $S := V(F, F_X, F_Y) \cap \mathbb{C}^n \times \mathbb{C}^2$ and $T := V(F_X, F_Y) \cap \mathbb{C}^n \times \mathbb{C}^2$ as submanifolds of $\mathbb{C}^n \times \mathbb{C}^2$. Then $S \cong \mathbb{C}^{n-1}$ is a hypersurface in $T \cong \mathbb{C}^n$. We shall study the holomorphic maps $\pi : \mathbb{C}^n \times \mathbb{C}^2 \to \mathbb{C}^n$ and $\pi_0 : S \to \mathbb{C}^n$ corresponding to the morphisms π and π_0 from above in the neighborhood of $\alpha \in \mathbb{C}^n$. We have

$$\pi_0^{-1}(\alpha) = \{\alpha\} \times \mathrm{Sing}(C_\alpha) = \{(\alpha, x_i, y_i) \,|\, i = 1, \dots, l\}.$$

Lemma 3.4. *Given $\delta > 0$ there are for small $\epsilon > 0$ open neighborhoods U_i of (α, x_i, y_i) on S $(i = 1, \dots, l)$ with the following properties:*

(i) The U_i are pairwise disjoint and

$$\pi_0^{-1}(U_\epsilon(\alpha)) = \bigcup_{i=1}^{l} U_i, \;\; U_i \subset U_\epsilon(\alpha) \times U_\delta(x_i, y_i) \text{ for } i = 1, \dots, l.$$

(ii) $\pi(U_i) \subset U_\epsilon(\alpha)$ is a submanifold of codimension 1 $(i = 1, \dots, l)$ and the map $\pi_0 : U_i \to \pi(U_i)$ is biholomorphic.

(iii) For any subset $\{j_1, \dots, j_\lambda\} \subset \{1, \dots, l\}$ with λ distinct elements $\pi(U_{j_1}) \cap \cdots \cap \pi(U_{j_\lambda})$ is a submanifold of $U_\epsilon(\alpha)$ of codimension λ.

Using the lemma we can finish the proof of Proposition 3.3 as follows: Since $\mathcal{H}_{pq}$ is open in $\mathcal{H}$ we can choose in 3.4 an $\epsilon > 0$ such that $U_\epsilon(\alpha) \cap \mathcal{H} \subset \mathcal{H}_{pq}$. Then for all $\beta \in U_\epsilon(\alpha) \cap \mathcal{H}$ it follows that C_β is a nodal curve of type p, q. By dimension reasons the set

$$B := \pi(U_{i_1}) \cap \cdots \cap \pi(U_{i_\lambda}) \setminus \bigcup_{i \notin \{i_1, \dots, i_\lambda\}} \pi(U_i)$$

is not empty. Moreover since the $U_i \subset U_\epsilon(\alpha) \times U_\delta(x_i, y_i)$ are pairwise disjoint and $\pi_0 : U_i \to \pi(U_i)$ is bijective, for any $\beta \in B$ the fiber $\pi_0^{-1}(\beta)$ consists of exactly λ points $(\beta, Q_k) \in U_{i_k}$ and $Q_k \in U_\delta(P_{i_k})$ for $k = 1, \dots, \lambda$. $\square$

Lemma 3.4. (i) We shall apply the Implicit Function Theorem to the map $(F_X, F_Y) : \mathbb{C}^n \times \mathbb{C}^2 \to \mathbb{C}^2$ given by F_X and F_Y. Remember that $S \cong \mathbb{C}^{n-1}$ is a hypersurface of $T = \{(\beta, x, y) \,|\, F_X(\beta, x, y) = F_Y(\beta, x, y) = 0\}$. The Jacobian of the map (F_X, F_Y) has rank 2 at the points (α, x_i, y_i) since the Hessian Hess_F is one of its 2-minors and $\mathrm{Hess}_F(\alpha, x_i, y_i) \neq 0$ for $i = 1, \dots, l$.

The Implicit Function Theorem states that there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ and holomorphic maps $\varphi_i : U_{\epsilon_0}(\alpha) \to U_{\delta_0}(x_i, y_i)$ with $\varphi_i(\alpha) = (x_i, y_i)$ such that $T \cap U_{\epsilon_0}(\alpha) \times U_{\delta_0}(x_i, y_i)$ is the graph $\Gamma_{\varphi_i} = \{(\beta, \varphi_i(\beta)) \,|\, \beta \in U_{\epsilon_0}(\alpha)\}$ of φ_i $(i = 1, \dots, l)$. The morphism π_0 of $\mathbb{C}$-schemes is finite. Then the underlying continuous map π_0 is closed with respect to the standard topology, as is well-known. Further $U := \bigcup_{i=1}^{l} S \cap \Gamma_{\varphi_i}$ is an open neighborhood of $\pi_0^{-1}(\alpha)$ on S. Hence $W := \mathbb{C}^n \setminus \pi_0(S \setminus U)$ is an open neighborhood of α in $\mathbb{C}^n$ such that $\pi_0^{-1}(W) \subset U$. For small $\epsilon \leq \epsilon_0$ we have $\pi_0^{-1}(U_\epsilon(\alpha)) \subset U$ and so

$$\pi_0^{-1}(U_\epsilon(\alpha)) = \bigcup_{i=1}^{l} U_i$$

where $U_i := S \cap (\Gamma_{\varphi_i} \cap \pi^{-1}(U_\epsilon(\alpha))) = S \cap \Gamma_{\varphi_i\,|\,U_\epsilon(\alpha)}$ is an open neighborhood of (α, x_i, y_i) on S $(i = 1, \ldots, l)$. For small $\epsilon > 0$, as $\varphi_1, \ldots, \varphi_l$ are continuous functions, the $U_1, \ldots, U_l$ are pairwise disjoint and $U_i \subset U_\epsilon(\alpha)) \times U_\delta(x_i, y_i)$ for $i = 1, \ldots, l$.
(ii) Since $U_i \subset \Gamma_{\varphi_i\,|\,U_\epsilon(\alpha)}$ is a submanifold of codimension 1 and $\pi : \Gamma_{\varphi_i\,|\,U_\epsilon(\alpha)} \to U_\epsilon(\alpha)$ is biholomorphic $\pi(U_i) \subset U_\epsilon(\alpha)$ is likewise a submanifold of codimension 1 and $\pi : U_i \to \pi(U_i)$ is biholomorphic.
(iii) The gradient of F at (α, x_i, y_i) has the form $(v_i, 0, 0)$ with $v_i := (\{x_i^\nu y_i^\mu\}_{\nu p + \mu q < pq})$ for $i = 1, \ldots, l$. By 2.1 the vectors v_i are linearly independent, and v_i is normal to the hypersurface $\pi(U_i)$ at α. It follows that $\pi(U_{i_1}) \cap \cdots \cap \pi(U_{i_\lambda})$ is for small $\epsilon > 0$ a submanifold of $U_\epsilon(\alpha)$ of codimension λ. $\square$

In connection with Theorem 3.2 we have a question: Given a Weierstraß semigroup close its greatest gap. Do we get again a Weierstraß semigroup?

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